Chapter 5 Magnetostatics

we learned that stationary charges produce static electric field: $\vec{\nabla} \times \vec{E} = 0$, $\vec{\nabla} \cdot \vec{D} = \rho$, $\vec{D} = \varepsilon \vec{E}$

Steady current produces magnetic field. when $\frac{\partial}{\partial t} = 0$ we have:

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{H} &= \vec{J} \end{aligned}}$$

J: current density

B: magnetic flux density

H: magnetic field intensity

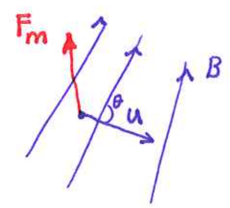
$B = \mu H$

For most dielectrics and metals: $\mu = \mu_0$ excluding ferromagnetic materials.

Magnetic Forces and Torques

A moving charge in a magnetic field experiences a force Fm:

$$\boxed{\vec{F}_m = q \vec{u} \times \vec{B}} \quad (N)$$

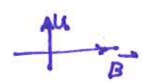


$F_m = quB \sin\theta$

The unit for B is Tesla (T) $\left(= \frac{N}{C \frac{m}{s}}\right)$

If $\theta = 0 \rightarrow F_m = quB \sin(0) = 0$

If $\theta = 90 \rightarrow F_m = quB$



If the charge particle is in the presence of both electric and magnetic field, the total electromagnetic force on the charge is:

$$\boxed{\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B} = q(\vec{E} + \vec{u} \times \vec{B})}$$ Lorentz Force

Note that:

1) $\vec{F}_e$ is always in the direction of $\vec{E}$. But $\vec{F}_m$ is always perpendicular to $\vec{B}$.

2) $\vec{F}_e$ acts on a charge particle whether or not it is moving. But $\vec{F}_m$ acts only on moving charge.

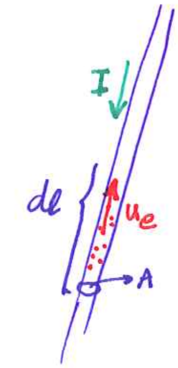
3) $\vec{F}_e$ expends energy in moving a charge, But Fm does not work when a particle is moved.

This is because Fm and u are normal: $\vec{F}_m \cdot \vec{u} = 0 \rightarrow$

$$d\vec{W} = \vec{F}_m \cdot \vec{dl} = \vec{F}_m \cdot \vec{u}\, dt = 0$$

Magnetic Force on a Current-Carrying Conductor

A current in a conductor consists of drifting charge particles. So a wire carrying a current in a magnetic field experiences a force, F_m. To find F_m, consider a segment of the wire.



$$d\vec{F}_m = dQ\,\vec{u}_e \times \vec{B}$$

$$= -Ne\,A\,dl\,\vec{u}_e \times \vec{B}$$

$dQ = \rho_{ve}\,dv = \rho_{ve}\,A\,dl$

$= -Ne\,A\,dl$

N: # of electrons per unit volume

$\vec{u}_e$ is parallel to $\vec{I}$ but in opposite direction. so we may write: $dl\,\vec{u}_e = -\vec{dl}\,u_e \Rightarrow$

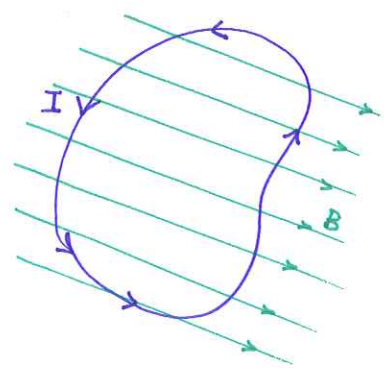
$$d\vec{F}_m = \underbrace{Ne\,A\,u_e}_{=I}\,\vec{dl} \times \vec{B} = I\,\vec{dl} \times \vec{B} \quad (N)$$

recall $\vec{J} = \rho\vec{u} = Ne\vec{u}_e$

$\vec{I} = \vec{J}A = Ne\vec{u}_e A$

For a closed circuit of current I, we have: $\boxed{\vec{F}_m = I\oint_C \vec{dl} \times \vec{B}}$

Closed Circuit in a Uniform $\vec{B}$ field



We want to calculate the total force on a closed wire carrying a current I in a uniform magnetic field B.

$$\vec{F}_m = I\oint_C \vec{dl} \times \vec{B}$$

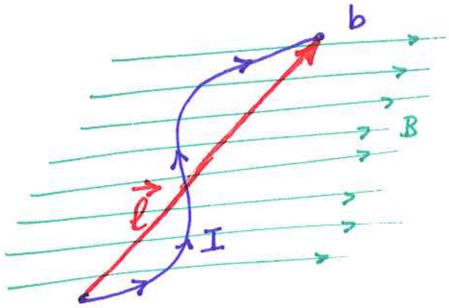
Since $\vec{B}$ is constant (uniform), it can be taken out from the integral:

$$\vec{F}_m = I\underbrace{\left(\oint_C \vec{dl}\right)}_{=0} \times \vec{B} = 0$$

So F_m on any closed circuit loop is zero in a uniform B.

Curved Wire in a Uniform B Field

We want to find the force on a wire segment in magnetic field as shown in the picture:



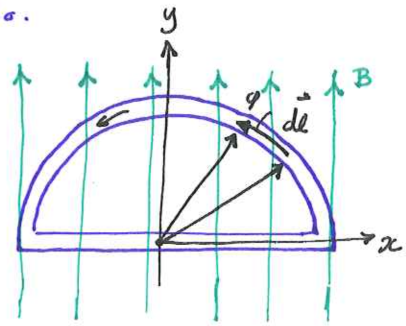
$$\vec{F}_m = I \int_a^b \vec{dl} \times \vec{B}$$

$$\vec{F}_m = I\left(\int_a^b \vec{dl}\right) \times \vec{B} = I \vec{l} \times \vec{B}$$ where $\vec{l}$ is a vector from a to b.

So F_m depends on the shortest distance between the points.

Example

A semicircular closed circuit wire is in a uniform magnetic field $\vec{B}$. It carrys a current I. Calculate the force on the straight section of the wire and the force on the curved section. The circuit is in x-y plane and $\vec{B} = \hat{y} B_o$.



- F_m on the straight section:

$$\vec{l} = \hat{x}\, 2r$$

$$\vec{F}_{m_1} = I \vec{l} \times \vec{B} = I\, 2r\, \hat{x} \times \hat{y}\, B_o = \hat{z}\, 2Ir\, B_o \quad (N)$$

- F_m on the circular section:

$$\vec{F}_{m_2} = I \int_{\phi=0}^{\pi} \vec{dl} \times \vec{B} = I \int_{\phi=0}^{\pi} r\, d\phi\, \hat{\phi} \times \hat{y}\, B_o = I r B_o \int_{\phi=0}^{\pi} -\hat{z} \sin\phi\, d\phi = -\hat{z}\, 2Ir\, B_o \quad (N)$$

The magnitude is same as the force on the straight section as expected.

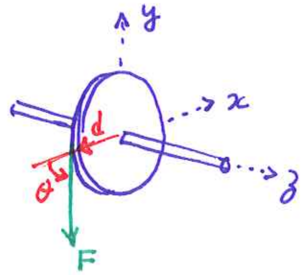
$$\vec{F}_{m_1} + \vec{F}_{m_2} = 0 \quad \checkmark$$

Magnetic Torque on a Current-Carrying Loop

When a force is applied on a pivoted body, it rotates about the fixed axis.

The torque for this rotation is given by:

$$\vec{T} = \vec{d} \times \vec{F} \quad (Nm)$$



For the disk in the picture:

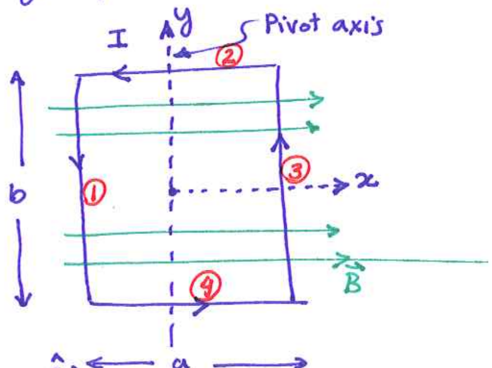
$$\vec{T} = d(-\hat{x}) \times (-\hat{y}) F$$
$$= \hat{z}\, r F$$

We may use the right-hand rule: { thumb points the direction of $\vec{T}$
four fingers show the direction of the rotation.

Magnetic Field in the Plane of the Loop

Let's calculate the torque on a rectangular conducting loop as shown in the picture.

$$\vec{B} = \hat{x} B_o$$

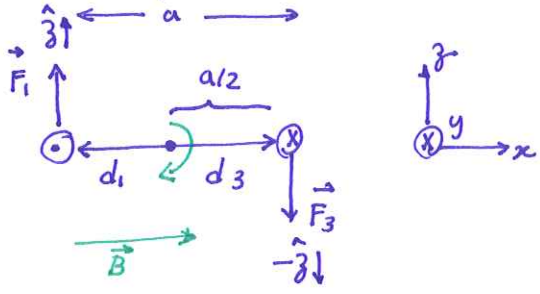


$$\vec{F}_1 = I\vec{\ell} \times \vec{B} = I(-b\hat{y}) \times (\hat{x} B_o) = \hat{z}\, I b B_o$$

$$\vec{F}_3 = I\vec{\ell} \times \vec{B} = I((b\hat{y}) \times (\hat{x} B_o)) = -\hat{z}\, I b B_o$$

on arms ② and ④ no force is applied because $\vec{\ell}$ and $\vec{B}$ are in palled:

$$\vec{F}_2 = \vec{F}_4 = I\vec{\ell} \times \vec{B} = I \ell B \sin(0) = 0$$

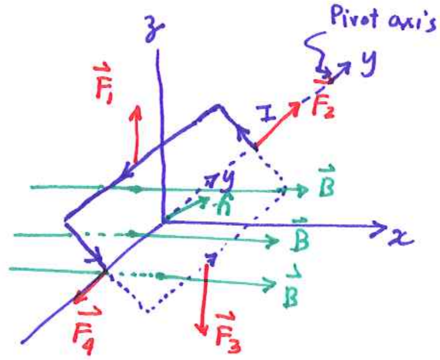


$$\vec{T} = \vec{T}_1 + \vec{T}_2 = \vec{d}_1 \times \vec{F}_1 + \vec{d}_2 \times \vec{F}_2$$

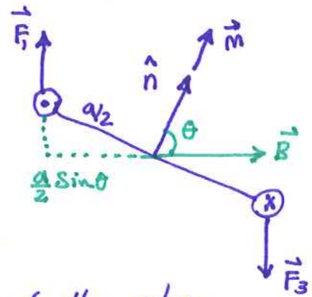
$$= \left(-\frac{a}{2}\hat{x}\right) \times (\hat{z}\, I b B_o) + \left(\frac{a}{2}\hat{x}\right) \times (\hat{z}\, I b B_o)$$

$$= \frac{1}{2} a I b B_o \hat{y} + \frac{1}{2} a I b B_o \hat{y} = \hat{y}\, I \overbrace{ab}^{A\,(\text{area})} B_o = \hat{y}\, I A B_o$$

Magnetic Field Perpendicular to the Axis of a Rectangular Loop:



Now consider that the plane of the rectangular loop is rotated so that its surface crosses the magnetic field lines $\vec{B}$. In this case the is an angle θ between the vector $\hat{n}$ (normal to the surface) and $\vec{B}$:



$$\vec{T} = \vec{d} \times \vec{F}$$

$$|T| = \frac{a}{2} \sin\theta \, F$$

Compared with the previous case we have an extra term of $\sin\theta$ that is associated with the rotation of the plane.

We can hence write:

$$T = I A B_0 \sin\theta$$

If the loop consists of N turns, the total torque is:

$$T = \underbrace{N I A}_{\equiv \text{ magnetic moment}, |\vec{m}|} B_0 \sin\theta$$

$\vec{m}$ is a vector in the direction of $\hat{n}$. obeys the right-hand rule:

- four fingers in the direction of current
- thumb specifies the direction of $\hat{n}$ and $\vec{m}$

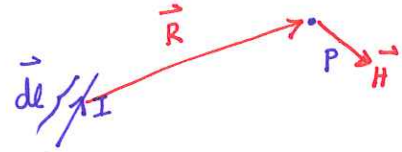
$$\boxed{\begin{array}{ll} \vec{m} \triangleq \hat{n} N I A & (A \cdot m^2) \\ \vec{T} = \vec{m} \times \vec{B} & (N \cdot m) \end{array}}$$

This expression is valid for any orientation of $\vec{B}$ and for a loop of any shape.

The Biot-Savart Law

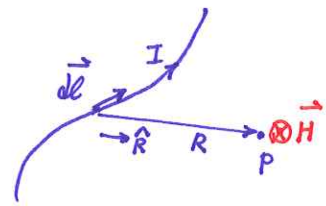
Jean Biot and Felix Savart derived an expression that relates the magnetic field intensity $\vec{H}$ to the current I that generates $\vec{H}$:

$$d\vec{H} = \frac{I}{4\pi}\,\frac{\vec{dl} \times \hat{R}}{R^2} \quad \left(\frac{A}{m}\right)$$



So for a conductor of a finite size we need to sum all the contributions:

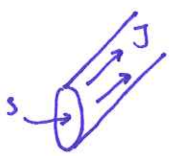
$$\boxed{\vec{H} = \frac{I}{4\pi}\int_l \frac{\vec{dl} \times \hat{R}}{R^2} \quad (A/m)}$$

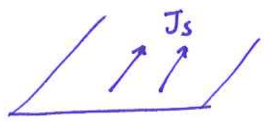


Magnetic Field due to Surface and Volume Current Distribution

If the current is ~~in~~ on a surface or in a volume, we may exted the above relation as:

$$I\vec{dl} = \vec{J}_s\,ds = \vec{J}\,dv$$





So we can write:

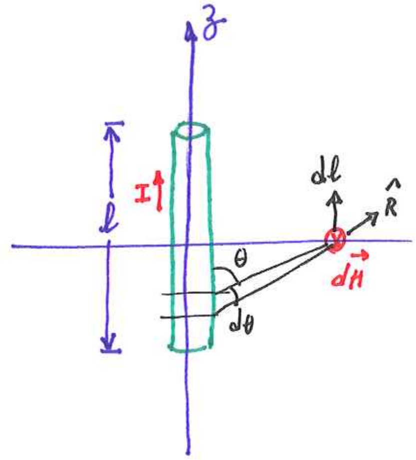
$$\vec{H} = \frac{1}{4\pi}\int_s \frac{\vec{J}_s \times \hat{R}}{R^2}\,ds \qquad \text{for a surface current}$$

$$\vec{H} = \frac{1}{4\pi}\int_V \frac{\vec{J} \times \hat{R}}{R^2}\,dv \qquad \text{for a volume current}$$

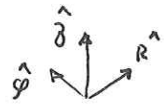
Example Magnetic Field of a Linear Conductor.

A linear conductor of length and carrying current I is placed along z-axis. Determine the magnetic flux $\vec{B}$ at point P at a distance r in the xy plane in free space.

$$\vec{H} = \frac{I}{4\pi}\int_{l} \frac{\vec{dl} \times \hat{R}}{R^2}$$



$$\vec{dl} \times \hat{R} = (dz\,\hat{z}) \times \hat{R} = \hat{\varphi}\,\sin\theta\, dz$$

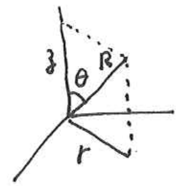


$$\rightarrow \vec{H} = \frac{I}{4\pi}\int_{z=-l/2}^{l/2} \hat{\varphi}\,\frac{\sin\theta}{R^2}\,dz$$

Replace: $R = r\csc\theta$ $\quad (r = R\cos\theta)$

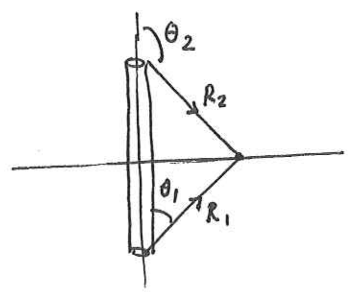
$z = -r\cot\theta$

$dz = r\csc^2\theta\, d\theta$



$$\rightarrow \vec{H} = \frac{I}{4\pi}\int_{\theta_1}^{\theta_2} \frac{r\csc^2\theta\, d\theta}{r^2\csc^2\theta}\, d\theta\, \hat{\Phi}$$

$$= \hat{\Phi}\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \hat{\Phi}\frac{I}{4\pi r}(\cos\theta_2 - \cos\theta_1)$$



$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2 + (\frac{l}{2})^2}} \qquad \cos\theta_2 = -\cos\theta_1 = \frac{-l/2}{\sqrt{r^2 + (\frac{l}{2})^2}}$$

$$\rightarrow \vec{B} = \mu_0\vec{H} = \hat{\Phi}\frac{\mu_0 I l}{4\pi r\sqrt{r^2 + (l/2)^2}} = \hat{\Phi}\frac{\mu_0 I l}{2\pi r\sqrt{4r^2 + l^2}}$$

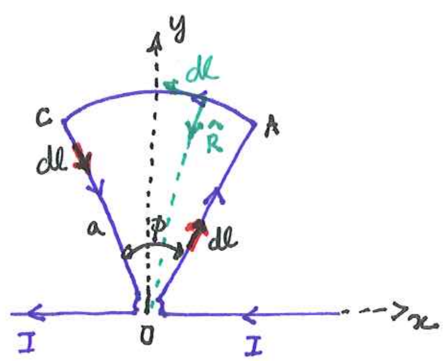
if $l \gg r \rightarrow \boxed{\vec{B} = \hat{\Phi}\frac{\mu_0 I}{2\pi r}}$ infinitely long wire

Example Find the magnetic field at the apex O of a pie-shaped loop as show below.

Segments OA and CA make a zero B at O ~~due to symmetry~~

because $\vec{dl}$ and $\vec{R}$ are parallel or anti-parallel:

$$\vec{dl} \times \hat{R} = 0 .$$



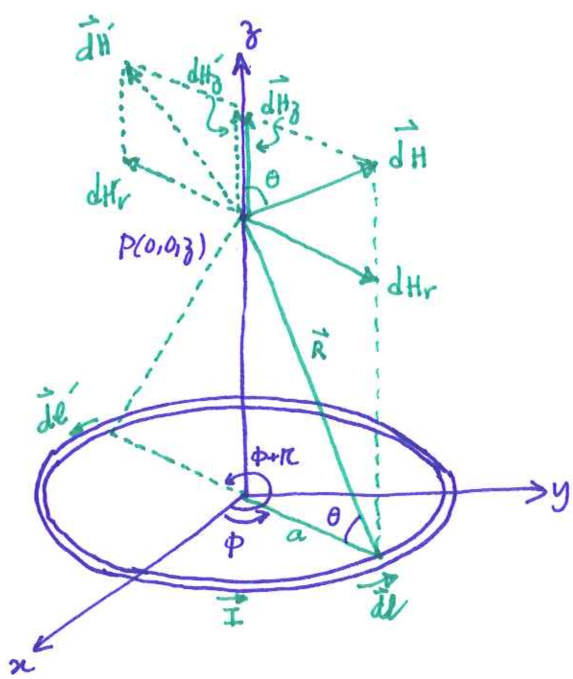
for segment AC, $\vec{dl}$ is perpendicular to $\hat{R}$ so:

$$\vec{dl} \times \hat{R} = \hat{z}\, dl = \hat{z}\, a\, d\varphi \Rightarrow$$

$$\vec{H} = \frac{I}{4\pi}\int \frac{\vec{dl} \times \hat{R}}{R^2} = \frac{I}{4\pi}\int \frac{\hat{z}\, a\, d\varphi}{a^2} = \hat{z}\,\frac{I}{4\pi a}\,\Phi$$

Example: Magnetic Field of a circular loop

A circular loop of radius r carries a steady current I. Determine $\vec{H}$ at a point on the axis of the loop.



$$dH = \frac{I}{4\pi \underbrace{R^2}_{(a^2+z^2)}} |\overbrace{\vec{dl} \times \hat{R}}^{dl}| = \frac{I\,dl}{4\pi(a^2+z^2)}$$

$\vec{dH}$ has components of dH_r and dH_z. Components of dH_r cancel each other. So:

$$\vec{dH} = \hat{z}\,dH_z = \hat{z}\,dH\cos\theta = \hat{z}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\,dl$$

$$\Rightarrow \vec{H} = \hat{z}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}\oint dl = \hat{z}\,\frac{I\cos\theta}{4\pi(a^2+z^2)}(2\pi a)$$

$$\cos\theta = \frac{a}{R} = \frac{a}{\sqrt{a^2+z^2}} \rightarrow \boxed{\vec{H} = \hat{z}\,\frac{Ia^2}{2(a^2+z^2)^{3/2}}} \quad \frac{A}{m}$$

At the center of the loop $(z=0)$: $\vec{H} = \hat{z}\,\frac{I}{2a}$

At points very far from the loop $z \gg a^2$: $\vec{H} = \hat{z}\,\frac{Ia^2}{2|z|^3}$

Magnetic Field of a Magnetic Dipole

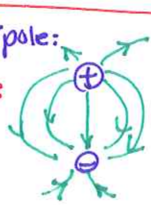
A loop of current has a magnetic moment $\vec{m} = \hat{z}m$ when the loop is in the x-y plane.
For this we have $m = IA = I\pi a^2 \Rightarrow \vec{H} = \hat{z}\,\frac{Ia^2}{2|z|^3} = \hat{z}\,\frac{m}{2\pi|z|^3}$ at $|z| \gg a$.
This is $\vec{H}$ at a point on z axis. If the point is a arbitrary point $P'(R', \theta', \phi')$:

$$\boxed{\vec{H} = \frac{m}{4\pi R'^3}\left(\hat{R}\,2\cos\theta' + \hat{\theta}\sin\theta'\right) \quad \text{when } R' \gg a}$$

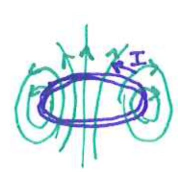
This field is called magnetic dipole.

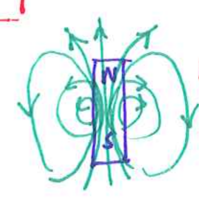
Compare with Electric dipole:

Electric dipole:



Magnetic dipole:





Bar magnet